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Propagation of Ocean Waves in Discrete Spectral Wave Models

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In many numerical models for hindcasting or forecasting ocean waves, wave energy is propagated over large distances. In the class of discrete spectral models such propagation suffers from a disintegration of the initial wave field into many individual wave fields. This "garden sprinkler" effect is due to the treatment of finite spectral bands as individual wave components. It is shown in the present study that this effect can be avoided by including two correction terms in the commonly used energy balance equation of the waves. One of these terms accounts for longitudinal (frequency) dispersion, the other term accounts for lateral (directional) dispersion. These terms are derived from the energy balance of finite spectral bands and they are expressed in terms of the spectral band characteristics. Since their nature is that of diffusion terms, they are local operators, which is computationally convenient. However, the coefficients of these terms are not locally determined. To illustrate the effect of the proposed correction terms, the propagation of swell from a distant storm (oceanic scale) is computed with and without the proposed correction terms. © 1987 Academic Press, Inc.

I. INTRODUCTION

Numerical computations of swell propagation over large distances across the ocean surface are routinely carried out in models for wave forecasting and wave hindcasting. In the linear approximation such computations are almost trivial for a single wave component. However, many of such models are basically formulated in terms of a discretized wave energy spectrum. This introduces nontrivial complications for the wave propagation computations since variations in propagation speed and direction in each spectral band introduce dispersion that presently operating models do not seem to handle properly.

Ideally, an initially narrow spatial distribution of wave energy on the ocean surface propagates in the model across the ocean while its horizontal extent increases linearly in time if the wave energy is distributed over a finite spectral bandwith. The energy of the entire spectrum will thus spread smoothly over the ocean. However, the spectral resolution in most of the presently operating models, in particular the directional resolution, is so coarse that a "garden sprinkler" effect occurs: an initial spatial distribution disintegrates into a number of similar distributions, one for each

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spectral band of the discretized spectrum. One consequence of this disintegration is that the arrival of swell from distant storms is often poorly predicted.

A straightforward solution to the above problem is to increase the spectral resolution. In fact, the spectral resolution should be such that the spreading of an initially narrow spatial distribution should not be larger than the mesh-size of the computational grid after the wave field has crossed the ocean. This implies that the width of the spectral bands in frequency (Δf) and direction ($\Delta\theta$) should fulfill the conditions

$$\Delta f < f/N$$

and

$$\Delta\theta < 1/N$$

in which N is the distance across the ocean expressed in number of meshes. A typical value for numerical wave models for the North Atlantic Ocean is $N = 35$, so that $\Delta f \simeq 0.03 f$ and $\Delta\theta \simeq 1.5°$. However, such a high directional resolution is not practical in many applications and other means of improvement are called for.

The above unsatisfactory state of affairs is discussed in general terms in a recent intercomparison study of ten advanced wave forecasting models [1]. For a review of relevant propagation algorithms reference is made to Isozaki and Uji [2], Resio, Garcia, and Vincent [3] and Young and Sobey [4]. The issue is adressed in detail by Gelci, Devillan, and Chavy [5] and by Greenwood and Cardone [6]. In these two publications two algorithms are described in which the propagated wave energy is redistributed at each grid point of the model over its immediate vicinity at each time step of the model. Gelci *et al.* [5] choose this redistribution explicitly such that the obtained spatial spreading of the wave energy as a function of time is some best approximation of the actual spreading of the waves during a period of three days. Another approach is suggested in the SWAMP study [1] in which a convolution filter is derived as a propagation operator for an initially narrow wave field (narrow in horizontal dimensions). However, the rationale for applying this operator to arbitrary wave fields is not given. Moreover, some numerical problems may be expected with this filter as its horizontal width in longitudinal direction is typically smaller than the grid-spacing in an ocean wave model (typically 50 km for the filter and 150 km for the grid-spacing).

In the present paper the solution to the problem is derived from the energy balance equation for arbitrary wave fields. The solution consists of adding two correction terms to the energy balance equation to represent spectral bands rather than spectral components. The effect of these terms is precisely what is required: they increase the spatial extent of an initial distribution linearly in time, they are local operators and their characteristics depend on the spectral band characteristics.

The structure of the paper is as follows. In Section II, we adress the theoretical aspects of our arguments. For reasons of clarity we restrict ourselves first to one-dimensional propagation. The arguments are later extended to two-dimensional propagation. The inclusion of the effects of wave generation and dissipation on the propagation scheme conclude our theoretical arguments. In Section III we compare results of near-exact wave propagation computations with the proposed computation method and two conventional methods. Some computational aspects are adressed in Section IV. Our conclusions are formulated in Section V.

II. Spectral Dispersion

A. *Introduction*

For most practical purposes, the wave field on the ocean can be described adequately with the two-dimensional energy density spectrum as a slowly varying function of time and space. The evolution of a single wave component in deep water with frequency f and direction of propagation θ can then be obtained from the energy balance equation [7, 8]

$$\frac{\partial}{\partial t} E(f, \theta; \mathbf{x}, t) + \nabla [\mathbf{c}(f, \theta) E(f, \theta; \mathbf{x}, t)] = S(f, \theta; \mathbf{x}, t). \tag{1}$$

This equation is valid for propagation in a Euclidean space, spanned by time t and space coordinates $\mathbf{x} = (x, y)$. E is the energy density of the wave component, $\mathbf{c}$ is the propagation speed vector and S is the sum effect of all processes of wave generation and wave dissipation. The left-hand side of this equation represents the propagation of the wave energy, so that for propagation problems we consider Eq. (1) with $S(f, \theta; \mathbf{x}, t) = 0$. The formulation of the energy balance equation in spherical coordinates to model wave propagation over a globe is touched upon in Section II.C.

Since the direction of propagation is constant in deep water, the wave energy density of one spectral wave component travels along straight lines, or if the earth's curvature is taken into account, along great circles, with a propagation speed given by linear wave theory. It has been established that this is true even for propagation over very large distances [9, 10, 11]. However, corrections are needed for the propagation of wave energy in finite spectral bands as considered in the present study.

B. *One-Dimensional Propagation*

We consider uni-directional waves propagating in the direction of a (one-dimensional) s-axis. The corresponding spectrum is then written as $E(f; s, t)$ and the propagation equation for individual wave components becomes

$$\frac{\partial}{\partial t} E(f; s, t) + \frac{\partial}{\partial s} [c(f) E(f; s, t)] = 0. \tag{2}$$

In conventional numerical wave models the propagation of finite spectral bands is considered rather than of individual components. We can obtain the corresponding energy propagation equation by integrating Eq. (2) over the frequency band considered. Denoting the central frequency of this band with f_i and the band width with Δf_i, the propagation equation becomes (dropping the notation for time and space)

$$\int_{f_i - 1/2\Delta f_i}^{f_i + 1/2\Delta f_i} \left[\frac{\partial}{\partial t} E(f) + \frac{\partial}{\partial s} c(f)\, E(f) \right] df = 0. \tag{3}$$

Defining $\bar{E}_i$ as the average energy density in this frequency band $(f_i - \frac{1}{2}\Delta f_i, f_i + \frac{1}{2}\Delta f_i)$, and changing the order of differentiation and integration in Eq. (3), we find

$$\frac{\partial}{\partial t} \bar{E}_i + \frac{\partial}{\partial s} \left[\frac{1}{\Delta f_i} \int_{f_i - 1/2\Delta f_i}^{f_i + 1/2\Delta f_i} c(f)\, E(f)\, df \right] = 0. \tag{4}$$

Now we consider the second term in this Eq. (4) separately. It seems acceptable, in view of other assumptions made in wave forecasting models, to approximate $c(f)$ and $E(f)$ as linearly dependent on f within the frequency band considered. If we define c_i as the propagation speed at frequency f_i, we can write

$$\frac{1}{\Delta f_i} \int_{f_i - 1/2\Delta f_i}^{f_i + 1/2\Delta f_i} c(f)\, E(f)\, df = c_i \bar{E}_i + \frac{1}{\Delta f_i} \int_{f_i - 1/2\Delta f_i}^{f_i + 1/2\Delta f_i} \{c(f) - c_i\}\{E(f) - \bar{E}_i\}\, df. \tag{5}$$

The first term on the right-hand side of this equation corresponds to the propagation of the energy density at the central frequency f_i. It is represented in every presently operating discrete spectral wave model. It then follows that the effect of the finite band width on the energy propagation is represented by the second term. Writing $\{c(f) - c_i\}$ as $c'_i(f - f_i)$ in which $c'_i = dc_i/df$, which is permitted if $c(f)$ depends linearly on f, we find that Eq. (4) becomes (inserting the notation for time and space and neglecting higher order terms):

$$\frac{\partial}{\partial t} \bar{E}_i(s, t) + \frac{\partial}{\partial s} \left[c_i \bar{E}_i(s, t) \right] + \frac{\partial}{\partial s} \left[\frac{\Delta f_i^2}{12} c'_i \frac{\partial}{\partial f} E(f; s, t) \right] = 0. \tag{6}$$

The first two terms are similar in nature to the terms on the left-hand side of the original energy balance equation (2). In fact, these two terms are often interpreted as representing the original propagation equation in numerical models. The third term of Eq. (6) provides the correction which is required to represent the effect of a finite band width.

The numerical implementation of the first two terms of Eq. (6) or (2) has been described in detail in the literature (see Introduction). It is therefore deemed unnecessary to elaborate on it here. The third term seems to be new and its numerical implementation is not obvious. A numerical approximation of the frequency derivative of the energy density $\partial E(f)/\partial f$ which appears in this term, would (for a conventional finite difference method) require the determination of E for frequences f_{i-1} and f_{i+1}. This does not seem to be a problem if the spectrum at any one time is smooth and broad compared with frequency band Δf_i. However, when swell propagates over large distances, its spectrum becomes rather narrow and the estimate of $\partial E(f)/\partial f$ is consequently inaccurate in most models. We therefore suggest here another approximation which relates information in the frequency domain to information in the space domain.

Consider the energy of a single wave component $E(f)$ at a time interval T after it has been imposed on the ocean surface at time t_0 (see Fig. 1). The spatial distribution propagates over the ocean undistorted since the energy is carried by one frequency only. We thus find

$$E(f;\, s, t) = E(f;\, \xi, t-T) \tag{7}$$

in which

$$\xi = s - c(f)\, T. \tag{8}$$

If for all frequencies in the frequency band $(f_i - \frac{1}{2}\Delta f_i, f_i + \frac{1}{2}\Delta f_i)$ an identical initial distribution is imposed at time $t_0 = t - T$, we find,

$$E(f;\, s, t) = \bar{E}_i(\xi, t-T). \tag{9}$$

The frequency derivative of the energy density can then be written, with Eq. (7) and (9) as

$$\frac{\partial}{\partial f} E(f;\, \xi, t-T) = \frac{\partial}{\partial \xi}\bar{E}_i(\xi, t-T)\frac{d\xi}{df} \tag{10}$$

and, with Eq. (8), it follows that

$$\frac{\partial}{\partial f} E(f;\, \xi, t-T) = -c_i' T \frac{\partial}{\partial \xi}\bar{E}_i(\xi, t-T). \tag{11}$$

With this result Eq. (6) can already be simplified considerably from a numerical

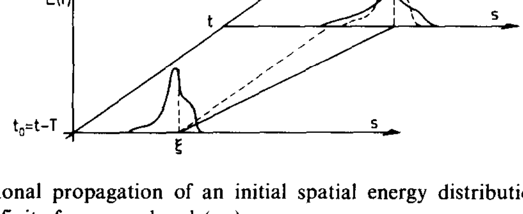

FIG. 1. One-dimensional propagation of an initial spatial energy distribution carried by one frequency (----) and by a finite frequency band (—).

point of view. A further simplification can be achieved if we assume $\partial \bar{E}_i/\partial \xi$ to be constant along the characteristic $dx/dt = c_i$, at least to first order:

$$\frac{\partial}{\partial \xi} \bar{E}_i(\xi, t-T) \simeq \frac{\partial}{\partial s} \bar{E}_i(s, t). \tag{12}$$

The error in this approximation of $\partial \bar{E}_i/\partial \xi$ is of second order in Δf and the term containing $\partial E/\partial f$ in the propagation equation (6) is a second order term in Δf, so that the error due to the above approximation is of fourth order in Δf. With equations (11) and (12) we can write Eq. (6) as

$$\frac{\partial}{\partial t} \bar{E}_i(s, t) + \frac{\partial}{\partial s}\left[c_i \bar{E}_i(s, t) \right] - \frac{\partial}{\partial s}\left[\frac{\Delta f_i^2}{12} c_i'^2 T \frac{\partial}{\partial s} \bar{E}_i(s, t) \right] = 0. \tag{13}$$

The third term in this Eq. (13) is the correction term to account for the finite spectral bandwith. It has the character of a diffusion term. We can write Eq. (13) also as

$$\frac{\partial}{\partial t} \bar{E}_i(s, t) + \frac{\partial}{\partial s}\left[c_i \bar{E}_i(s, t) \right] - \frac{\partial}{\partial s}\left[D_i \frac{\partial}{\partial s} \bar{E}_i(s, t) \right] = 0. \tag{14}$$

The time-dependent diffusion coefficient D_i is then

$$D_i = \Delta c_i^2 \, T/12 \tag{15}$$

in which Δc_i denotes the propagation speed difference across the frequency band $(f_i - \frac{1}{2}\Delta f_i, f_i + \frac{1}{2}\Delta f_i)$,

$$\Delta c_i = c_i' \, \Delta f_i \tag{16}$$

and T the time elapsed since the energy was imposed on the ocean surface, or, in terms of wave generation, the time elapsed since the energy was generated instantaneously by the wind. Such instantaneous generation does not in fact occur on the ocean so that a procedure should be defined to assign a value to T. We return to this problem in Section II.D.

To demonstrate that the correction term in Eq. (14) has the desired property to spread an initial spatial distribution linearly in time at the proper rate, we compare here an exact calculation of the propagation of a continuous spectrum with the proposed propagation of its discretized version. Consider a δ-function in space, i.e., a spatial distribution with a horizontal dimension which is small in terms of propagation distances but sufficiently large in terms of wavelength to exploit the notion of wave spectra and to ignore the dispersion of leading and trailing edges. It is located at position $s=0$ at time $t=0$ as the initial spatial distribution of wave energy that is distributed evenly over an arbitrary but narrow frequency band

$(f_i - \frac{1}{2}\Delta f_i, f_i + \frac{1}{2}\Delta f_i)$. For a continuous spectrum the energy density at frequency f is propagating with speed $c(f)$ so that at time t,

$$E(f;\, s, t) = E(f;\, s - c(f)\, t, t). \tag{17}$$

The highest frequency in the frequency band has propagated a distance $(c - \frac{1}{2}\Delta c)\, t$ while the lowest frequency has propagated a distance $(c + \frac{1}{2}\Delta c)\, t$. The total wave energy at time t is thus uniformly distributed over the interlying ocean area. The exact solution for the width of this block function, expressed in terms of its standard deviation σ_1, is

$$\sigma_1 = \Delta ct/\sqrt{12}. \tag{18}$$

The discrete version of the continuous spectrum is one carrier frequency $f_i = f$ with the assigned frequency band width $\Delta f_i = \Delta f$. The analytical solution of Eq. (14) for the above initial δ-distribution is the propagating Gauss-distribution (Fig. 2),

$$\bar{E}_i(s, t) = 1/(t\sqrt{2\pi\alpha}) \exp - \left[\frac{(s - c_i t)^2}{2\alpha t^2}\right] \tag{19}$$

in which α is $\Delta c_i^2/12$. The peak value of this distribution, equal to $1/(t\sqrt{2\pi\alpha})$, is located at $s = c_i t$. The standard deviation σ_2 of this distribution is

$$\sigma_2 = \Delta c_i t/\sqrt{12}. \tag{20}$$

Comparing Eqs. (18) and (20) it is obvious that the correction term in Eq. (14) increases the width of the spatial energy distribution linearly in time and that it does so at the appropriate rate.

The correction term in the propagation equation (14) has then all the desired qualities: (a) it is derived from the energy balance equation of the waves, (b) it is formulated in terms of the spectral band width, which is gratifying from a fundamental point of view; (c) it spreads an initial distribution linearly in time at the appropriate rate; (d) being a diffusion term it is a local operator, which is computationally convenient.

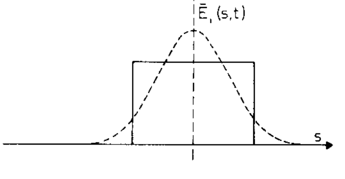

FIG. 2. One-dimensional propagation of an initial δ-function carried by a frequency band, as a block function, exact solution (—) and as a Gauss-distribution, approximated solution (----). The standard deviation of both functions increases proportionally with time and frequency bandwidth.

C. *Two-Dimensional Propagation*

The derivation of the two-dimensional propagation equation for a discretized spectrum is analogous to that of the one-dimensional version (Eq. (14)), except for a directional correction term. Following the same procedure as in Section II.B we find as equivalent of Eq. (6),

$$\frac{\partial}{\partial t}\bar{E}_{ij}+\frac{\partial}{\partial x}\left[c_{ijx}\,\bar{E}_{ij}+\frac{\Delta f_i^2}{12}c'_{ijx}\frac{\partial}{\partial f}E(f,\theta)+\frac{\Delta\theta_j^2}{12}c^*_{ijx}\frac{\partial}{\partial\theta}E(f,\theta)\right]$$
$$+\frac{\partial}{\partial y}\left[c_{ijy}\bar{E}_{ij}+\frac{\Delta f_i^2}{12}c'_{ijy}\frac{\partial}{\partial f}E(f,\theta)+\frac{\Delta\theta_j^2}{12}c^*_{ijy}\frac{\partial}{\partial\theta}E(f,\theta)\right]=0 \tag{21}$$

in which $c'_{ijx}=\partial c_{ijx}/\partial f$ and $c^*_{ijy}=\partial c_{ijy}/\partial\theta$ and the average energy density

$$\bar{E}_{ij}=\frac{1}{\Delta f_i\Delta\theta_j}\int_{f_i-1/2\Delta f_i}^{f_i+1/2\Delta f_i}\int_{\theta_j-1/2\Delta\theta_j}^{\theta_j+1/2\Delta\theta_j}E(f,\theta)\,df\,d\theta. \tag{22}$$

The subscripts x and y refer to x- and y-component, i and j are number indices in f- and θ-space, respectively.

The frequency derivative of the energy density $\partial E(f,\theta)/\partial f$ can be treated as before but the direction derivative $\partial E(f,\theta)/\partial\theta$ needs to be treated differently. Analogously to Eq. (7) and (8) we assume the energy density to be constant along a characteristic. If in addition we assume again identical spatial distribution for all components in a spectral bin$(f_i-\frac{1}{2}\Delta f_i,\ f_i+\frac{1}{2}\Delta f_i;\ \theta_j-\frac{1}{2}\Delta\theta_j,\ \theta_j+\frac{1}{2}\Delta\theta_j)$ then

$$E(f,\theta;x,y,t)=\bar{E}_i(f,\theta;\xi_x,\xi_y,t-T) \tag{23}$$

in which

$$\xi_x=x-c_xt,\qquad \xi_y=y-c_yt. \tag{24}$$

Using Eq. (23) and (24) and approximating $\partial E/\partial\xi_x$ and $\partial E/\partial\xi_y$ at time $t-T$ with $\partial E/\partial x$ and $\partial E/\partial y$ at time t, the direction derivative of the energy density can be approximated by

$$\frac{\partial}{\partial\theta}E(f,\theta;x,y,t)\simeq -c_iT\sin\theta_j\frac{\partial}{\partial x}\bar{E}_{ij}(x,y,t)+c_iT\cos\theta_j\frac{\partial}{\partial y}\bar{E}_{ij}(x,y,t). \tag{25}$$

It is now convenient to use polar coordinates (n,s) rather than the cartesian coordinates (x,y)

$$x=s\cos\theta_j-n\sin\theta_j,\qquad y=s\sin\theta_j+n\cos\theta_j. \tag{26}$$

The frequency and direction derivatives of $E(f, \theta; x, y, t)$ can then be written as

$$\frac{\partial}{\partial f} E(f, \theta; x, y, t) \simeq -c_i' T \frac{\partial}{\partial s} \bar{E}_{ij}(n, s, t) \tag{27}$$

and

$$\frac{\partial}{\partial \theta} E(f, \theta; x, y, t) \simeq c_i T \frac{\partial}{\partial n} \bar{E}_{ij}(n, s, t). \tag{28}$$

Substituting polar coordinates and these expressions for $\partial E(f, \theta; x, y, t)/\partial f$ and $\partial E(f,\theta; x, y, t)/\partial \theta$ in Eq. (21) results in the final two-dimensional energy balance equation of the discretized spectrum,

$$\frac{\partial}{\partial t} \bar{E}_{ij}(n, s, t) + \frac{\partial}{\partial s} [c_i \bar{E}_{ij}(n, s, t)] - \frac{\partial}{\partial s} \left[D_{ss} \frac{\partial}{\partial s} \bar{E}_{ij}(n, s, t) \right] - \frac{\partial}{\partial n} \left[D_{nn} \frac{\partial}{\partial n} \bar{E}_{ij}(n, s, t) \right] = 0 \tag{29}$$

in which

$$D_{ss} = \Delta c_i^2 T/12 \tag{30}$$

and

$$D_{nn} = c_i^2 \Delta\theta_j^2 T/12. \tag{31}$$

Equation (29) is the main result of this study. It differs from the conventionally used equation of wave propagation (1) in that two correction terms are included: one for longitudinal frequency dispersion and another for transversal directional dispersion. The equation can be readily rewritten in cartesian coordinates, using tensor conventions,

$$\frac{\partial}{\partial t} \bar{E}_{ij}(x, y, t) + \frac{\partial}{\partial x} \left[c_{ijx} \bar{E}_{ij}(x, y, t) - D_{xx} \frac{\partial}{\partial x} \bar{E}_{ij}(x, y, t) \right] + \frac{\partial}{\partial y} \left[c_{ijy} \bar{E}_{ij}(x, y, t) - D_{yy} \frac{\partial}{\partial y} \bar{E}_{ij}(x, y, t) \right] - 2D_{xy} \frac{\partial^2}{\partial x\, \partial y} \bar{E}_{ij}(x, y, t) = 0 \tag{32}$$

in which

$$D_{xx} = D_{ss} \cos^2\theta + D_{nn} \sin^2\theta, \tag{33}$$

$$D_{yy} = D_{ss} \sin^2\theta + D_{nn} \cos^2\theta, \tag{34}$$

$$D_{xy} = (D_{ss} - D_{nn}) \sin\theta \cos\theta. \tag{35}$$

The above derivation of the energy balance equation for spectral bands is based on propagation over a flat earth (Euclidean coordinates). Swell propagating on a global scale is obviously affected by the curvature of the earth's surface so that for such a situation the derivation should be given in spherical coordinates. This would modify the above proposed directional dispersion term such that an initially narrow wave field (δ-function in x, y space) carried by one frequency f_i and a finite directional band $\Delta\theta_j$ spreads initially in time but starts to decrease its horizontal dimensions when it has travelled a distance equal to one quarter of the earth's circumference. The spatial standard deviation of the propagating energy distribution is (compare with Eq. (19) and (20)),

$$\sigma = \Delta\theta_j R \sin(c_i t/R)/\sqrt{12} \tag{36}$$

in which R is the radius of the earth. This implies that when the wave component has travelled over a distance of half the earth's circumference ($c_i t = \pi R$, the distance to the antipodean point), the value of σ is zero and the initial δ-function is again a δ-function. Such a spreading can be achieved with the following value of D_{nn},

$$D_{nn} = c_i \Delta\theta_j^2 \, R \sin(c_i T/R) \cos(c_i T/R)/12. \tag{37}$$

In addition to this change in D_{nn} in Euclidean coordinates, the modified energy balance equation (32) should be transformed to spherical coordinates latitude and longitude. The resulting equation is somewhat similar to Eq. (32) except that one extra term appears. This term accounts for the change of propagation direction relative to true North when travelling along a great circle (see Introduction).

No attempt is made here to carry out this transformation. The determination of the time T elapsed since the waves were imposed on the ocean surface is trivial if the wave field is imposed on the ocean surface at one moment. It is not so trivial in a more realistic situation where waves are continuously generated. This aspect is adressed next.

D. *Wave Generation and Dissipation*

The model as presented above applies to waves which are generated instantaneously and which propagate without any supply or withdrawal of energy. In practical applications the waves grow or decay continuously in time. Strictly speaking one would need to propagate the generated energy over the entire ocean for every time step in the model in which wave energy is generated. This is obviously impractical; one would want to use one propagation model in which the correction term is based on some integral time lapse instead of the time span T which was considered above. We define here an integral time lapse τ as the "age" of the wave component (f, θ) based on the development of that component in the past. Reference is made to the Appendix for details. Only the results are mentioned here. The wave age τ is a wave property which propagates with the wave energy and for which we can write an evolution equation

$$\frac{d}{dt}\tau(f, \theta; \mathbf{x}, t) = 1 - \frac{S_g(f, \theta; \mathbf{x}, t)}{E(f, \theta; \mathbf{x}, t)}\tau(f, \theta; \mathbf{x}, t) \tag{38}$$

in which $S_g(f, \theta; \mathbf{x}, t)$ is equal to the sum of the generation terms in the source function $S(f, \theta; \mathbf{x}, t)$. We assumed that wave dissipation does not affect the wave age.

The evolution equation (38) is simple in form and readily interpreted. Obviously if waves propagate without growth or decay, i.e., $S_g = 0$, wave age must increase linearly in time,

$$\frac{d}{dt}\tau(f, \theta; \mathbf{x}, t) = 1. \tag{39}$$

The generation of "new" wave energy rejuvenates the waves and, as indicated in Eq. (38), the wave age decreases proportionally with the rate of adding new energy. Since we have assumed in deriving equation (38) that wave decay affects "old" and "young" wave energy to the same degree, the total wave age is not affected by decay. Energy dissipation is therefore not represented in the evolution equation (38). We can write the left-hand side of this evolution equation as an operator which is identical to that for the propagation of wave energy, including the proposed correction terms (Eq. (32)). The advantage is that the increase in computing effort to include wave age in the model is relatively small. In fact, the extra effort is only to compute, store and retrieve the simple expression on the right hand side of the evolution equation (38).

We wish to emphasize that the concept of "wave age" introduced here should not be confused with the traditional concept of wave age which is usually a nondimensional number indicating the degree of development of the wave field, e.g., the ratio of the phase speed at the peak frequency of the wave spectrum and the local wind speed [12], or a dimensionless peak frequency [13].

An alternative to the above model for a time-dependent wave age is to use a constant, characteristic value for the wave age (thus avoiding adding equations to the model). For instance, Gelci *et al.* [5] developed a model in which the energy is redistributed in x, y space at every time step in the model. This model amounts to a model with constant diffusion coefficients, or, in terms of this study, a constant wave age. The spatial spreading of energy is consequently proportional to the square root of time instead of linear with time as it should be. Gelci *et al.* [5] chose the redistribution coefficients such that the spreading in the model is some best fit [redacted] shown that for an initial spatial δ-function the wave age should be taken as $T/2$. However, this choice implies a fair amount of error in the results prior to time T. If one wishes to decrease that difference, a smaller wave age could be taken. But a constant wave age of $T/3$ (as suggested in Fig. 11 of [5]) would, in the case of the propagating Gauss distribution of Eq. (19), still result in an error of 18% at time $t = T$ in the standard deviation of the spatial distribution. In the model with the time-dependent wave age an error of this magnitude would only be obtained if that wave age were constantly underestimated by 33% (which seems to be an unrealistically large underestimate for any well-behaved wave model).

Obviously, for a model with a constant wave age that is used in situations with continuous generation and dissipation of wave energy, the value of T should be chosen such that it corresponds to some characteristic distance between the wave generating storms in the area of propagation and the point(s) of forecast.

III. Illustrations

A. *Introduction*

To demonstrate the effect of the proposed corrections we compare results of typical model computations (low spectral resolution) with results of computations that closely approximate exact propagation (high spectral resolution). Since our basic concern in this study is the propagation of waves and not their generation or dissipation, we impose the wave field instantaneously on the ocean surface. We ignore wave generation and wave dissipation thereafter during the propagation. The illustrations are given for the spatial distribution of the swell part of the spectrum since the relevant effects are thus best demonstrated.

Some numerical propagation algorithms unintentionally contain so-called numerical diffusion, simply as a result of the type of finite-difference approximation used for the nondiffusive transport terms in the original energy balance (Eq. (1)). Others do not contain such numerical diffusion or only to a small degree. Since the proposed correction terms are of a diffusive nature, we consider numerical diffusion as a separate parameter in the illustrations.

We thus distinguish four situations:

(1) Computations that closely approximate exact propagation. These are carried out for the original propagation equation (1), without numerical diffusion and with a high spectral resolution.

(2) Computations carried out with commonly used low spectral resolution for

(a) the original equation (1), without numerical diffusion,

(b) the original propagation equation (1), with numerical diffusion as present in many operational models,

(c) the corrected propagation equation (32), with negligible numerical diffusion.

B. *Methods of Computation*

The two computations based on the original propagation equation (1) and carried out without numerical diffusion (situations (1) and (2a)) are performed with a simple back-tracing technique. We take the value of the energy density of the wave component (f, θ) at location (x, y) at time t equal to its value at the appropriate location at time $t=0$,

$$E(f, \theta; x, y, t) = E(f, \theta; x - c_x t, y - c_y t, 0). \tag{40}$$

This is the exact solution of the wave energy propagation equation (1) in the absence of dissipation or generation of wave energy for a single spectral component (f, θ). The difference between situations (1) and (2a) arise only as a result of different spectral resolution. The values for Δf and $\Delta\theta$ are 0.005 Hz and 5°, respectively, for situation (1) and 0.02 Hz and 30°, respectively, for situation (2a).

For the other situations ((2b) and (2c)) we model the original propagation equation (1) with an algorithm that is virtually free from numerical diffusion and to which we add diffusion terms of our choice. For situation (2b) we choose to add isotropic diffusion terms with constant coefficients to simulate numerical diffusion. For situation (2c) the added terms are the proposed correction terms given in Eq. (32). The algorithm used is an explicit predictor–corrector algorithm that can be considered as an iterative approximation of the Crank–Nicholson scheme [14]. To assess the propagation errors of this algorithm we consider the situation of the propagating Gauss-distribution which is described in Section II.B (Eq. (19)). The propagation was computed with this algorithm with the original propagation equation (1) to which the proposed correction terms were added. The error in the spreading of the distribution was less than 2% for conditions that are almost equal to those in situations (2b and 2c), see below. We are therefore confident that the algorithm is virtually free of numerical diffusion for the situations considered below and also that it reproduces the effects of the added diffusion terms properly. The spectral resolution for the computations of situations (2b) and (2c) is the same as for situation (2a), $\Delta f = 0.02$ Hz and $\Delta\theta = 30°$.

C. *Initial Conditions*

We consider the propagation of swell from a distant storm in an area of 4200 × 3000 km (roughly the dimensions of the North Atlantic Ocean) covered with a grid of 150 × 150 km mesh-size. The time step in the numerical computations is 1.5 hr. These are values which are fairly commonly used in presently operating numerical hindcast or forecast models. The period over which the computation extends is 90 hr.

The initial wave field in the storm is represented as a Gaussian distribution of wave energy in x, y space with a standard deviation of 200 km centred at location $x = 600$ and $y = 600$ km (Fig. 3(a)). The initial wave field in each grid point in the storm area is represented with a two-dimensional spectrum $E(f, \theta)$ of the Pierson–Moskowitz type [15] with a $\cos^2(\theta)$-directional distribution:

$$E(f, \theta) = \frac{2}{\pi} \alpha g^2 (2\pi)^{-4} f^{-5} \exp\left[-\frac{5}{4} \left(\frac{f}{f_p} \right)^{-4} \right] \cos^2(\theta - \theta_0). \tag{41}$$

The peak frequency f_p was chosen as 0.1 Hz. In the centre of the storm the value of α was chosen as 0.016 (significant wave height about 5.5 m). We choose the initial main wave direction $\theta_0 = 30°$ so as not to have the main direction of propagation coincide with the direction of one of the axes of the grid. The initial value of the transported quantity $\bar{E}_{ij}$ is determined from Eq. (41) according to the definition in Eq. (22) with a resolution which is 1/5 of the resolution that is used in the propagation computations (both in frequency and direction, see above).

D. *Results*

The effect of including the proposed correction terms can be well illustrated with the spatial distribution of wave energy that has travelled a long distance undisturbed by wave generation of dissipation. Such energy is usually low-frequency energy (swell) and we define it here as $E_{0.1}$,

$$E_{0.1} = \int_0^{2\pi} \int_0^{0.1\ \mathrm{Hz}} E(f, \theta)\, df\, d\theta. \tag{42}$$

The results of the near-exact computations (situation (1)) are given in Fig. 3(b) for time $t = 90$ hr. The initially narrow distribution (Fig. (3a)) has spread into a smooth, longitudinally narrow (~ 750 km half power width) and laterally broad (~ 3500 km half-power width) distribution over roughly a quarter circle section at a distance of about 3000 km from the source area. That these results are a good approximation of the exact situation was checked by repeating the computations with increased spectral resolution (half the values of Δf and $\Delta \theta$). The changes in the results were negligible.

The effect of reducing the spectral resolution to the commonly used resolution of $\Delta f = 0.02$ Hz and $\Delta \theta = 30°$ is well illustrated by comparing the above results of situation (1) with the results of the low-resolution computations of situation (2a) (no numerical diffusion, Fig. 3c). The disintegration of the initial wave field into a number of individual wave fields is obvious. The results for situation (2b) which illustrate the presence of numerical diffusion that is inadvertedly present in many algorithms, are given in Fig. 3d. For this situation we choose to simulate an isotropic diffusion with a magnitude between that of a Lax scheme and that of an upwind difference scheme [14]. The values of the corresponding diffusion coefficients are $\Delta x^2/2\Delta t$ ($\simeq 2 \times 10^6$ m^2/s in our case) and $c\Delta x/2$ ($\simeq 0.75 \times 10^6$ m^2/s in our case), respectively. We choose $D_{nn} = D_{ss} = 10^6$ m^2/s as a typical value. The wave energy, after 90 hr of propagation, is too widely distributed over the ocean in this case and its maximum value is too low (see Fig. 3d). The longitudinal extent of the

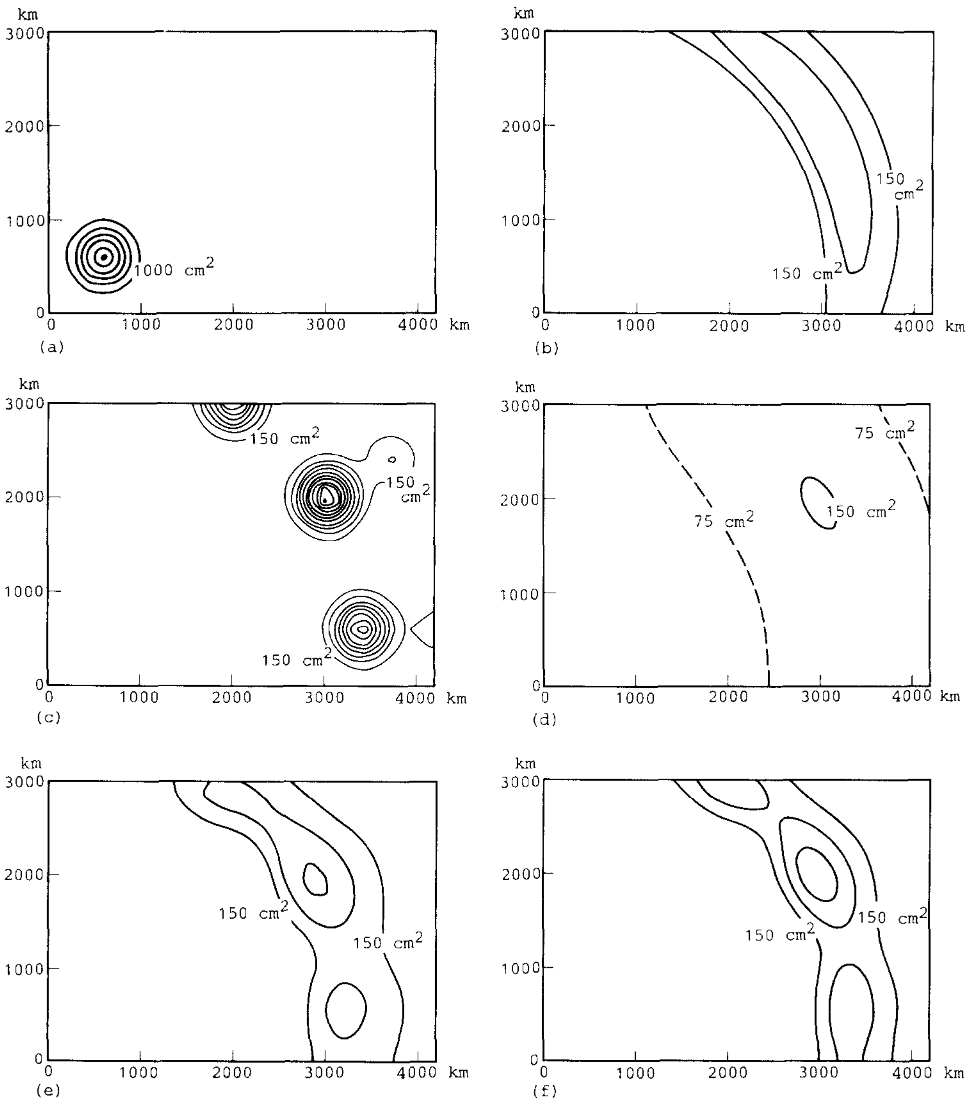

FIG. 3. Contourline plots of swell energy $E_{0.1}$ distribution over the ocean according to various propagation models after 90-hr travel time. The contourline interval is 150 cm^2 except in panel (a) where it is 1000 cm^2 and in panel (d) where an intermediate level is shown. Panel (a) initial distribution, panel (b) near-exact solution (situation (1)), panels (c) and (d) propagation without proposed correction (situation (2a) low resolution; no numerical diffusion and (situation (2b) with numerical diffusion respectively), panels (e) and (f) propagation with proposed correction, (situation (2c) low resolution with corrected propagation, $\Delta x = \Delta y = 150$ km and $\Delta x = \Delta y = 75$ km, respectively).

distribution is now approximately 2000 km whereas it is 750 km in the near-exact computations (situation (1)). The lateral extent is also too large, but its magnitude cannot be determined from the illustration.

The proposed corrections (situation (2c)) improve the situation considerably [Fig. 3e]. Comparison with Fig. 3b (near-exact computation) shows that the spreading of the initial wave field seems to be adequately modelled for many practical problems. But one unexpected error is noticeable in this case. The location of the wave field is too close to the source area by about 10% of the expected distance. This is due to an inherent error of the chosen numerical algorithm. The ratio of the mesh-size in the model (= 150 km) to the length of the initial distribution ($\simeq$ 800 km) is large enough to under-estimate the propagation speed by 10% to 20% (e.g., [14]). Decreasing the mesh-size to 75 km, and the time step to 1 hr, removed this discrepancy (Fig. 3*f*). We also used a constant wave age of 45 hr (wave age = $T/2$, see Section II.D) to see whether the resulting spatial distribution at 90 hr would differ from the one in Fig. 3e. As expected, the differences were marginal.

IV. Computational Aspects

We consider here the following computational aspects of the proposed propagation method

(a) implementation,

(b) spectral resolution,

(c) propagation scheme.

(a) *Implementation*

The extra effort required to compute the proposed corrections seems rather modest. It pertains mostly to calculating the correction terms and the corresponding wave age. If we assume that the source function S is computationally simple, then adding two constant anisotropic diffusion terms might imply an increase in total computing time of 20% to 30%. The wave age model would add another 20 to 30% if the similarity between the wave age evolution equation and the wave energy balance equation is exploited. If the source term is computationally complicated, then the extra computer time to compute the proposed corrections is relatively small (marginal). The computer storage requirements however increase considerably, since the amount of data to be stored almost doubles.

(b) *Spectral Resolution*

The spectral resolution of $\Delta f = 0.02$ Hz and $\Delta\theta = 30°$, that we used in the illustrations and which is fairly characteristic for most wave forecasting models, implies that the longitudinal spreading is much smaller than the transversal spreading. Such directionality in the propagation characteristics of the model seems to be unbalanced. One would prefer to have a more or less isotropic spreading. This is achieved by taking the longitudinal diffusion coefficient D_{ss} equal to the transversal diffusion coefficient D_{nn}. This implies the following relationship between the directional resolution ($\Delta\theta$) and the frequency resolution (indicated by Δc):

$$\Delta\theta = \Delta c/c \qquad (\theta \text{ in radians}). \tag{43}$$

As $\Delta\theta$ is a constant in wave models, the relevant parameter for the frequency resolution is the relative difference of propagation speed across on frequency band. This is equivalent in deep water to the relative band width $\Delta f/f$ itself. In our illustrations (with a Pierson–Moskowitz spectrum) a resolution of $\Delta f = 0.02$ Hz provided sufficiently accurate results for waves with frequencies of about 0.1 Hz, i.e., a frequency resolution of $\Delta f = 0.2f$. However, for the more narrow JONSWAP spectrum [16] we suggest that a resolution of $\Delta f = 0.1f$ would be adequate for most purposes for models operating on a scale of the North Atlantic Ocean. This normalization of the frequency resolution implies a resolution along the frequency axis that is rather high at low frequencies and rather coarse at high frequencies. This seems to agree with a requirement that the shape of the spectrum should be well resolved. Most wind wave spectra require a higher resolution in the swell dominated low-frequency part than in the wind sea dominated high-frequency part with its f^{-5} equilibrium tail. It follows from the condition of isotropic spreading (eq. (43)) that a frequency resolution of $\Delta f = 0.1$ to $0.2f$ corresponds to a directional resolution $\Delta\theta \simeq 5°$ to $10°$ for every frequency. Obviously, a model without the proposed corrections would require a much higher resolution to obtain similar results as a model with the proposed corrections (if numerical diffusion is absent). It should be pointed out that if the ideal resolution mentioned in the Introduction is used, the proposed corrections are not needed ($\Delta f = f/N$ and $\Delta\theta = 1/N$).

(c) *Propagation Scheme*

In our numerical experiments we used a predictor–corrector propagation scheme that can be interpreted as an iterative Crank–Nicholson scheme. This method suited our purpose well because it can be shown to be free from numerical diffusion. However, for very high values of the diffusion coefficient in the correction term (which were not attained in the given illustrations), the scheme proved to be unstable. This is characteristic for all explicit propagation schemes. We therefore recommend to use a scheme that has negligible numerical diffusion and that is also unconditionally stable and efficient such as the Alternating Direction Implicit method. Other methods with appreciable numerical diffusion such as forward-time, central-space methods or the fully implicit method of Laasonen are not recommended. Properties of various numerical schemes are described in such text books as [14, 17, 18].

V. Conclusions

To represent the propagation of waves across the ocean in discrete spectral wave models one should use the energy balance equation of spectral bands rather than of individual wave components. Economic considerations force certain limitations on the band width so that a coarse spectral resolution cannot be avoided. To make the best of such a situation one can use the energy balance equation of an individual wave component corrected for the finite spectral band width. This correction can be approximated with two correction terms which have the character of diffusion terms. One of these accounts for longitudinal (frequency) dispersion and the other accounts for transversal (directional) dispersion. The two terms proposed in this study have the following desired qualities:

(i) they are derived from the energy balance equation of the waves;

(ii) their characteristics depend on the spectral band characteristics;

(iii) they spread an initially narrow spatial distribution of wave energy linearly in time at the proper rate;

(iv) they are local operators, although the diffusion coefficients are nonlocal.

The proposed correction terms depend on the wave age which is the time lapsed since the generation of the wave energy considered. A pragmatic definition is suggested for situations with a continuous growth and decay of wave energy.

Numerical experiments illustrate that using the proposed correction terms improves the propagation characteristics of the model considerably as compared with conventional models. Computations with the proposed correction terms based on the notion of wave age are estimated to require about 50% (or less) more computer time and about 100% more computer storage capacity than conventional computations. This is far less than would be required for an increase in spectral resolution having similar qualities of representing the continuous wave field. An even more economic alternative is also suggested.

It is recommended to use an Alternating Direction Implicit method for the propagation scheme since this is an unconditionally stable scheme without numerical diffusion. For models operating on oceanic scales with the proposed corrections a frequency resolution of $\Delta f = 0.1$ to $0.2f$ (or smaller) and a directional resolution of $\Delta\theta = 5$ to 10° (or smaller) is recommended.

APPENDIX: Wave Age

To arrive at a definition of an integral time lapse τ to replace the time span T, we consider the wave energy density $E(f, \theta; \mathbf{x}, t)$ to be the sum of energy contributions from the past. Such a contribution at time $t = t_g$ is denoted with $F(f, \theta; \mathbf{x}, t; t_g)$ so that the energy density at time t can accordingly be written as

$$E(f, \theta; \mathbf{x}, t) = \int_{-\infty}^{t} F(f, \theta; \mathbf{x}, t; t_g)\, dt_g \tag{A1}$$

in which $F(f, \theta; \mathbf{x}, t; t_g)\, dt_g$ is equal to the growth of $E(f, \theta; \mathbf{x}, t)$ in the time interval dt_g at time $t = t_g$. If we consider the source function S as the sum of the generative processes (denoted by S_g) and the dissipative processes (denoted by S_d),

$$\frac{d}{dt}[E(f, \theta; \mathbf{x}, t)] = S_g(f, \theta; \mathbf{x}, t) + S_d(f, \theta; \mathbf{x}, t) \tag{A2}$$

then $F(f, \theta; \mathbf{x}, t; t_g)$ is equal to $S_g(f, \theta; \mathbf{x}, t_g)$ at time $t = t_g$. We define wave age τ as the weighted average of the time lapse $(t - t_g)$ with the contribution $F(f, \theta; \mathbf{x}, t; t_g)$ as weight (dropping f and θ from the notation):

$$\tau(\mathbf{x}, t) \equiv \frac{1}{E(\mathbf{x}, t)} \int_{-\infty}^{t} F(\mathbf{x}, t; t_g)(t - t_g)\, dt_g. \tag{A3}$$

The rate of change of τ in a frame of reference moving with the wave energy is found by considering the derivative of the product $E(\mathbf{x}, t) \cdot \tau(\mathbf{x}, t)$,

$$\begin{aligned} \frac{d}{dt}[E(\mathbf{x}, t) \cdot \tau(\mathbf{x}, t)] &= \frac{d}{dt} \int_{-\infty}^{t} F(\mathbf{x}, t; t_g) \cdot (t - t_g)\, dt_g \\ &= \int_{-\infty}^{t} F(\mathbf{x}, t; t_g)\, dt_g + \int_{-\infty}^{t} \frac{d}{dt} F(\mathbf{x}, t; t_g)(t - t_g)\, dt_g. \end{aligned} \tag{A4}$$

The first term on the right-hand side is equal to $E(\mathbf{x}, t)$ (see Eq. (A1)); the second term can be evaluated by assuming that the decay of $F(f, \theta; \mathbf{x}, t; t_g)$ is proportional to $F(f, \theta; \mathbf{x}, t; t_g)$ itself,

$$\frac{d}{dt} F(\mathbf{x}, t; t_g) = \frac{S_d(\mathbf{x}, t)}{E(\mathbf{x}, t)} \cdot F(\mathbf{x}, t; t_g) \qquad \text{for } t > t_g \tag{A5}$$

It follows from Eqs. (A1)–(A4) that

$$\begin{aligned} \frac{d}{dt}[E(\mathbf{x}, t) \cdot \tau(\mathbf{x}, t)] &= E(\mathbf{x}, t) + \frac{S_d(\mathbf{x}, t)}{E(\mathbf{x}, t)} \int_{-\infty}^{t} F(\mathbf{x}, t; t_g) \cdot (t - t_g)\, dt_g \\ &= E(\mathbf{x}, t) + S_d(\mathbf{x}, t) \cdot \tau(\mathbf{x}, t). \end{aligned} \tag{A6}$$

When the product rule of differentiation is applied to Eqs. (A2) and (A6) one finds that

$$\frac{d}{dt} \tau(\mathbf{x}, t) = 1 - \frac{S_g(\mathbf{x}, t)}{E(\mathbf{x}, t)} \tau(\mathbf{x}, t) \tag{A7}$$

which is the evolution equation for the wave age τ defined in Eq. (A3).

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